

Inflation in a Symmetric Vacuum

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If in a finite universe, the tree-level vacuum is a symmetric superposition of coherent states, in each of which the inflaton assumes a different, energy-minimizing mean value (vev), then the resulting energy is positive and decreases exponentially as the volume of the universe increases. This effect can drive inflation when that volume is small and explain part of dark energy when it is big, but the effect is exceedingly tiny except at very early times.

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I. THE SOMBRERO

If we quantize the fields of the standard model in the temporal gauge $A_b^0(x) = 0$, then the vacuum—and all physical states—are invariant under time-independent gauge transformations. So we can imagine that the tree-level vacuum of a finite universe is a symmetric superposition of coherent states, in each of which a scalar inflaton assumes a different mean value (vev) that minimizes the inflaton potential (and the other fields vary accordingly). The resulting energy then is positive, approaching zero as the volume of the universe increases.

In this paper, I evaluate this effect for the simpler case of global gauge transformations. Integrating once over the brim of the sombrero, I find that the effect can drive inflation when the universe is tiny and explain a small part of dark energy when it is huge, but that the effect is exceedingly small except at very early times.

I discuss coherent-state vacua in Sec. II and show that the effect vanishes exponentially with the volume of the universe. I describe the symmetric coherent vacua and compute their energies for theories with respectively reflection symmetry, U(1) symmetry, and SU(2) symmetry in Secs. III, IV, and V.

II. COHERENT STATES

In what follows, we will focus on a single mode of the inflaton—the zero-momentum mode. In terms of this mode, a single real field ϕ of mass m at time $t = 0$ is

$$\phi(x) = \phi = \left(\frac{1}{2mV}\right)^{1/2} (a + a^\dagger) \quad (1)$$

and its conjugate momentum is

$$\pi(x) = \pi = \left(\frac{m}{2V}\right)^{1/2} (-ia + ia^\dagger). \quad (2)$$

A coherent state [1] is an eigenstate of the annihilation operator a with eigenvalue α

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (3)$$

The mean values of ϕ and π in this state then are

$$\langle\alpha|\phi|\alpha\rangle = \left(\frac{1}{2mV}\right)^{1/2} (\alpha + \alpha^*) \quad (4)$$

and

$$\langle\alpha|\pi|\alpha\rangle = \left(\frac{m}{2V}\right)^{1/2} (-i\alpha + i\alpha^*). \quad (5)$$

We will be considering theories in which the hamiltonian contains both π^2 and an inflaton potential U that vanishes for non-zero values ϕ_0 of the field. In such theories, a vacuum value of α must be real

$$\alpha_0 = x_0 \quad (6)$$

with

$$\phi_0 = \langle\alpha_0|\phi|\alpha_0\rangle = \langle x_0|\phi|x_0\rangle = \left(\frac{2}{mV}\right)^{1/2} x_0 \quad (7)$$

and $\langle x_0|\pi|x_0\rangle = 0$. Thus the argument x_0 of a vacuum coherent state

$$x_0 = \left(\frac{mV}{2}\right)^{1/2} \phi_0 \quad (8)$$

is proportional to the square-root of the volume of the universe.

The inner product of two coherent states is

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta^*\alpha\right), \quad (9)$$

which shows that they are normalized to unity. When both $\alpha = x$ and $\beta = x'$ are real, this inner product is

$$\langle x'|x\rangle = \exp\left(-\frac{1}{2}(x - x')^2\right). \quad (10)$$

When they also are arguments of vacuum coherent states, their inner product by (8)

$$\langle x'|x\rangle = \exp(-mV(\phi_0 - \phi'_0)^2/4) \quad (11)$$

vanishes exponentially fast as the volume of the universe expands. Thus in a spatially infinite universe, these states and the Hilbert spaces built upon them are orthogonal.

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III. REFLECTION SYMMETRY

Suppose the inflaton potential is

$$U = g : (\phi^2 - \phi_0^2)^2 : \quad (12)$$

in which the colons denote normal ordering, that is, a 's appear to the right of all a^\dagger 's. There are then two vacuum coherent states $|\pm x_0\rangle$ with x_0 given by (8) for which the mean value of the potential vanishes

$$\langle x_0|U|x_0\rangle = \langle -x_0|U|-x_0\rangle = 0. \quad (13)$$

Under the reflection $\phi \rightarrow -\phi$, the states $|\pm x_0\rangle$ are interchanged. The symmetric state

$$|Sx_0\rangle = N(|x_0\rangle + |-x_0\rangle) \quad (14)$$

is invariant under a reflection and is normalized when

$$N = \left[2 \left(1 + e^{-2x_0^2}\right)\right]^{-\frac{1}{2}}. \quad (15)$$

Because the matrix elements $\langle \pm x_0 | : \phi^n : | \mp x_0 \rangle$ vanish when n is a non-negative integer, the mean value of the inflaton potential in the symmetric state $|Sx_0\rangle$ is

$$\langle Sx_0|U|Sx_0\rangle = \frac{g\phi_0^4}{(1 + e^{mV\phi_0^2})}. \quad (16)$$

This energy is huge when $mV\phi_0^2$ is tiny and tiny when $mV\phi_0^2$ is huge. So although unrealistic, it has the qualitative form to drive inflation in a baby universe and to accelerate the expansion in a mature universe. Unfortunately, the epoch during which $mV\phi_0^2$ is small is very brief, and the contribution to dark energy when $mV\phi_0^2$ is big is exponentially suppressed.

IV. U(1) SYMMETRY

We now consider the more realistic case of a complex inflaton $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ made as usual of two real fields of equal mass. We'll take the inflaton potential to be

$$U_1 = g : (\phi^\dagger \phi - \phi_0^2)^2 : \quad (17)$$

in which $\phi_0 > 0$. Now the vacuum coherent states are direct-product states $|x_1, x_2\rangle$ with

$$x_1^2 + x_2^2 = \left(\frac{mV}{2}\right) (\phi_{10}^2 + \phi_{20}^2) = mV\phi_0^2. \quad (18)$$

Their inner products by (10) are

$$\langle x'_1, x'_2 | x_1, x_2 \rangle = e^{-(x_1 - x'_1)^2/2} e^{-(x_2 - x'_2)^2/2}. \quad (19)$$

We may denote these states by an angle θ

$$|x_1, x_2\rangle = |\theta\rangle \quad (20)$$

with

$$x_1 + ix_2 = \sqrt{mV} \phi_0 e^{i\theta}. \quad (21)$$

In terms of angular variables, the inner product (19) then is

$$\langle \theta' | \theta \rangle = \exp \left\{ -2mV\phi_0^2 \sin^2[(\theta - \theta')/2] \right\}. \quad (22)$$

The inflaton vacuum $|S\phi_0\rangle$

$$|S\phi_0\rangle = N_1 \int_0^{2\pi} d\theta |\theta\rangle \quad (23)$$

is symmetric; it is normalized when

$$N_1 = 1/\sqrt{8\pi W(y_0)} \quad (24)$$

where $W(y)$ is

$$W(y) = \int_0^{\pi/2} dx \exp(-y \sin^2 x) \quad (25)$$

and $y_0 = 2mV\phi_0^2$. The integral formula

$$\int_0^{\pi/2} dx \sin^{2n} x = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \quad (26)$$

lets us write $W(y)$ as the power series

$$W(y) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} \frac{(2n-1)!!}{(2n)!!}. \quad (27)$$

The complex field ϕ is

$$\phi = \left(\frac{1}{2\sqrt{mV}} \right) \left[a_1 + a_1^\dagger + i(a_2 + a_2^\dagger) \right] \quad (28)$$

and so some of its coherent-state matrix elements are

$$\begin{aligned} \langle \theta' | : (\phi^\dagger)^n \phi^m : | \theta \rangle &= \langle \theta' | \theta \rangle \phi_0^{n+m} \left(\frac{e^{-i\theta'} + e^{-i\theta}}{2} \right)^n \\ &\times \left(\frac{e^{i\theta'} + e^{i\theta}}{2} \right)^m. \end{aligned} \quad (29)$$

The matrix element of the inflaton potential thus is

$$\langle \theta' | U_1 | \theta \rangle = g \langle \theta' | \theta \rangle \phi_0^4 \sin^4[(\theta - \theta')/2]. \quad (30)$$

The mean value of this potential in the symmetric state $|S\phi_0\rangle$ then is

$$\langle S\phi_0 | U_1 | S\phi_0 \rangle = g\phi_0^4 W(y_0)^{-1} W''(y_0). \quad (31)$$

For $y \ll 1$, we may use the power series (27) to write

$$W(y) \approx \frac{\pi}{2} \left(1 - \frac{y}{2} + \frac{3y^2}{16} - \frac{5y^3}{96} + \frac{35y^4}{3072} + \dots \right) \quad (32)$$

and

$$W''(y) \approx \frac{\pi}{16} \left(3 - \frac{5}{2}y + \frac{35y^2}{32} + \dots \right). \quad (33)$$

So to lowest order in $y_0 = 2mV\phi_0^2 \ll 1$, the mean value of the inflaton potential in the symmetric state $|S\phi_0\rangle$ is

$$\langle S\phi_0|U_1|S\phi_0\rangle \approx \frac{3g}{8}\phi_0^4 \left(1 - \frac{2}{3}mV\phi_0^2 \right), \quad (34)$$

which might drive inflation in an early, tiny universe while $mV\phi_0^2$ is small. But that epoch is very brief.

In the other limit, $y \rightarrow \infty$, we may use the approximations

$$W(y) \approx \int_0^\infty e^{-yx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{y}} \quad (35)$$

and

$$W''(y) \approx \int_0^\infty x^4 e^{-yx^2} dx = \frac{3}{8y^2}\sqrt{\frac{\pi}{y}} \quad (36)$$

to estimate the dark-energy density as

$$\langle S\phi_0|U_1|S\phi_0\rangle \approx \frac{3g}{16m^2V^2} \quad (37)$$

as $y_0 = 2mV\phi_0^2 \rightarrow \infty$. This energy density would be a negligibly small contribution to dark energy.

V. SU(2) SYMMETRY

The inflaton of the standard model has two complex components

$$h = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \quad (38)$$

in which ϕ_i are real scalar fields of mass m . The inflaton potential is then

$$U_2 = g : (h^\dagger h - \phi_0^2)^2 :. \quad (39)$$

The vacuum coherent states now are direct-product states $|x_1, x_2, x_3, x_4\rangle$ whose arguments lie on the surface of a three-dimensional sphere of radius $\sqrt{mV}\phi_0$ in four-space:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= \left(\frac{mV}{2} \right) (\phi_{10}^2 + \phi_{20}^2 + \phi_{30}^2 + \phi_{40}^2) \\ &= mV (|h_1|^2 + |h_2|^2) = mV\phi_0^2. \end{aligned} \quad (40)$$

By an extension of Eq. (19), their inner product is

$$\langle x'_1, x'_2, x'_3, x'_4 | x_1, x_2, x_3, x_4 \rangle = e^{-(x-x')^2/2} \quad (41)$$

in which

$$(x - x')^2 = \sum_{i=1}^4 (x_i - x'_i)^2. \quad (42)$$

In Hopf's coordinates,

$$\begin{aligned} x_1 + ix_2 &= (\sqrt{mV}\phi_0) e^{i\theta_1} \sin \eta \\ x_3 + ix_4 &= (\sqrt{mV}\phi_0) e^{i\theta_2} \cos \eta \end{aligned} \quad (43)$$

the symmetric state $|S_2\phi_0\rangle$ is

$$|S_2\phi_0\rangle = N_2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{\pi/2} d\eta \sin 2\eta |\theta_1, \theta_2, \eta\rangle \quad (44)$$

and the inner product (41) is

$$\begin{aligned} \langle \theta'_1, \theta'_2, \eta' | \theta_1, \theta_2, \eta \rangle &= \exp \{ -mV\phi_0^2 [1 \\ &\quad - \cos(\theta_1 - \theta'_1) \sin \eta \sin \eta' \\ &\quad - \cos(\theta_2 - \theta'_2) \cos \eta \cos \eta'] \}. \end{aligned} \quad (45)$$

The normalization constant N_2 is

$$N_2 = 1 / \left(2\pi \sqrt{W_2(y_0)} \right) \quad (46)$$

where $W_2(y)$ is the integral

$$\begin{aligned} W_2(y) &= e^{-y} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{\pi/2} d\eta \int_0^{\pi/2} d\eta' \sin 2\eta \sin 2\eta' \\ &\quad \times e^{y \cos \theta_1 \sin \eta \sin \eta'} e^{y \cos \theta_2 \cos \eta \cos \eta'} \end{aligned} \quad (47)$$

and $y_0 = mV\phi_0^2$.

Some of the coherent-state matrix elements of normally ordered monomials of the inflaton components are

$$\begin{aligned} \langle \theta'_1, \theta'_2, \eta' | : (h_1^\dagger)^n h_1^j (h_2^\dagger)^k h_2^\ell : | \theta_1, \theta_2, \eta \rangle &= \phi_0^{n+j+k+\ell} \\ &\times \langle \theta'_1, \theta'_2, \eta' | \theta_1, \theta_2, \eta \rangle \left(\frac{e^{-i\theta_1} \sin \eta + e^{-i\theta'_1} \sin \eta'}{2} \right)^n \times \\ &\left(\frac{e^{i\theta_1} \sin \eta + e^{i\theta'_1} \sin \eta'}{2} \right)^j \left(\frac{e^{-i\theta_2} \cos \eta + e^{-i\theta'_2} \cos \eta'}{2} \right)^k \\ &\times \left(\frac{e^{i\theta_2} \cos \eta + e^{i\theta'_2} \cos \eta'}{2} \right)^\ell. \end{aligned} \quad (48)$$

So the corresponding matrix element of the inflaton potential (39) is

$$\begin{aligned} \langle \theta'_1, \theta'_2, \eta' | U_2 | \theta_1, \theta_2, \eta \rangle &= \frac{\phi_0^4}{4} \langle \theta'_1, \theta'_2, \eta' | \theta_1, \theta_2, \eta \rangle \times \\ &[1 - \cos(\theta_1 - \theta'_1) \sin \eta \sin \eta' - \cos(\theta_2 - \theta'_2) \cos \eta \cos \eta']^2 \end{aligned} \quad (49)$$

So the mean value of the inflaton potential (39) in the symmetric vacuum (44) is

$$\langle S_2\phi_0 | U_2 | S_2\phi_0 \rangle = \frac{g\phi_0^4}{4} \frac{W_2''(y_0)}{W_2(y_0)}. \quad (50)$$

We may expand the quartic integral (47) in powers of y , do the integrals over θ_1, θ_2, η , and η' , and write $W_2(y)$

as the double sum

$$W_2(y) = 8\pi^2 e^{-y} \sum_{j,k=0}^{\infty} \frac{j!(2j-1)!!}{(2j)!(2j)!!} \frac{k!(2k-1)!!}{(2k)!(2k)!!} \frac{y^{2j+2k}}{(j+k+1)!} \quad (51)$$

which is useful in the $y \rightarrow 0$ limit where

$$W_2(y) \approx 8\pi^2 e^{-y} \left(1 + \frac{y^2}{4} + \frac{y^4}{96} \right). \quad (52)$$

So in an early, tiny universe the vacuum energy is

$$\langle S_2 \phi_0 | U_2 | S_2 \phi_0 \rangle \approx \frac{3g\phi_0^4}{8} \left(1 - \frac{2mV\phi_0^2}{3} + \frac{(mV\phi_0)^3}{9} \right). \quad (53)$$

Unfortunately, the period during which $mV\phi_0^2 < 1$ is very brief—it is when the radius of the universe was small compared to $1/m$ and to $1/\phi_0$.

In the $y \rightarrow \infty$ limit,

$$W_2(y) \approx \frac{16\pi\sqrt{2\pi}}{y^{5/2}} \quad (54)$$

and so the dark-energy density is

$$\langle S_2 \phi_0 | U_2 | S_2 \phi_0 \rangle \approx \frac{35}{16} \frac{g}{m^2 V^2}, \quad (55)$$

which would be a negligibly small contribution to dark energy.

VI. CONCLUSION

A globally gauge-invariant coherent-state vacuum might explain inflation in the very early universe, but I can't imagine a model in which this effect could cause 40 e-foldings; it could contribute to dark energy in a big universe, but only negligibly.

In this paper, I have considered for simplicity only global gauge transformations and the zero-momentum mode of the inflaton. By path-integrating over all time-independent gauge transformations in a model with a more realistic inflaton [2], one might arrive at an effect that numerically is more impressive.

[1] R. J. Glauber, *Phys. Rev. Letters* **10(3)**, 84 (1963).

[2] R. Allahverdi, B. Dutta, and A. Mazumdar, *Phys. Rev.*

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